Axial quasi-normal modes of neutron stars: Accounting for the superfluid in the crust

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We present the results of the first study of global oscillations of relativistic stars with both elastic crusts and interpenetrating superfluid components. For simplicity, we focus on the axial quasi-normal modes. Our results demonstrate that the torsional crust modes are essentially unaffected by the coupling to the gravitational field. This is as expected since these oscillations are known to be weak gravitational-wave sources. In contrast, the presence of a loosely coupled superfluid neutron component in the crust can have a significant effect on the oscillation spectrum. We show that the entrainment between the superfluid and the crust nuclei is a key parameter in the problem. Our analysis highlights the need for a more detailed understanding of the coupled crust-superfluid at the microphysical level. Our numerical results have, even though we have not considered magnetised stars, some relevance for efforts to carry out seismology based on quasi-periodic oscillations observed in the tails of magnetar flares. In particular, we argue that the sensitive dependence on the entrainment may have to be accounted for in attempts to match theoretical models to observational data.

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1   Introduction

Following the observations of quasi-periodic oscillations (QPOs) in the tails of giant flares in several soft gamma-ray repeaters (SGRs) [1, 2, 3], axial (torsional) oscillations of neutrons stars have attracted considerable interest. The SGRs are generally thought to be highly magnetised neutron stars (magnetars) [4], and it would be natural to explain QPOs in the range $\sim 30 - 150$ Hz as axial crustal oscillations. That magnetar activity might trigger such oscillations was, in fact, suggested quite some time ago [5]. This explanation is supported by the spacing of the QPO frequencies [6]. The model does, however, face a serious challenge since the strong magnetic field should couple the crust to the core in less than an oscillation period [7, 8]. In addition, the magnetic field will affect the motion of the crust itself [9, 10, 11]. To determine global "elasto-magnetic" oscillation modes is challenging due to the possible existence of an Alfvén continuum in the core. The presence of this continuum casts doubt on the very existence of global mode solutions with a discrete frequency spectrum, see [12, 13] for recent progress on the purely fluid problem. However, recently there has been some evidence [14, 15] in favour of the hypothesis put forward in [8]. For moderate magnetic fields ($B \lesssim 10^{15}$ G) the global modes most easily excited by a catastrophic event in the crust have frequencies

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that are tuned to the purely elastic crustal mode frequencies. For very high magnetic fields $\sim 10^{16}$ G the situation is not quite as clear [16].

Most available studies of the axial mode problem have completely ignored the dynamical role of the dripped neutrons in the inner crust. Since these neutrons are likely to be superfluid they can flow through the crust lattice. This leads to the problem having an additional degree of freedom, and it is important to establish to what extent the presence of this new component will affect the seismology. A recent local analysis [17] has shown that the superfluid components, both in the crust and the core, may have significant effects. The main aim of this paper is to consider this problem in more detail. We will, for the first time, examine the effects of the superfluid crust component on the global axial oscillation spectrum. The obtained results have immediate relevance for the magnetar discussion. We are also reporting real progress in modelling the dynamics of realistic neutron stars. Our discussion provides insight into the global dynamics of solids coexisting with a superfluid component, and highlights the microphysics parameters that are needed if we want to improve the analysis. As we will see, the entrainment between the superfluid neutrons and the crust nuclei (the "effective mass" of the free neutrons) is a key parameter that needs to be constrained better by equation of state calculations.

Our analysis is based on state-of-the-art matter modelling. We employ a linear perturbation scheme in general relativity, and do not make additional approximations such as the Cowling approximation, wherein the gravitational degrees of freedom are neglected (or, as discussed in [6], the coupling is ignored). Since we account for the dynamics of spacetime itself, the oscillation modes we determine can be split into two distinct families, the torsional t-modes associated with the matter motion supported by the elastic properties of the solid, and the gravitational w-modes (which in turn can be subdivided into different classes, see $e.g.$ [18]). In principle, we should be able to address directly the gravitational-wave damping of the t-modes and the influence of the elastic and superfluid nature of the matter on the w-modes. In practice, however, we are unable to compute the damping time of the t-modes. This result is obvious if we estimate the coupling between the torsional motion of a solid and the gravitational field. Using the quadrupole formula, Schumaker & Thorne [19] estimate the gravitational-wave damping time scale of the t-modes to be $\sim 10^4$ years, $i.e.$ some fourteen orders of magnitude longer than the typical oscillation time-scale for a 30 Hz mode. The upshot of this is that the complex angular frequency ω in the standard ansatz for the time-dependence $e^{i\omega t}$ has an imaginary part which is fourteen orders of magnitude smaller than the real part. This is a problem for any numerical scheme that aims at determining the complex ω. The, perhaps naive, numerical scheme we use here is certainly unable to compute the imaginary part of these very slowly damped modes. We are, however, able to compute the real part, and hence the frequencies, of the t-modes accurately. The obtained results confirm the expectation that these modes can be accurately determined within the Cowling approximation [6]. This is true regardless of whether the superfluid neutrons are taken into account or not. The free neutrons on the other hand do affect the axial mode frequencies, in agreement with the qualitative results in [17]. The effect on the global mode frequencies turns out to be at the 10% level, but depends strongly on the precise properties of the inner crust. Overall, our results imply that, as far as seismology efforts are concerned [6], the Cowling approximation should be sufficient. We will return to that problem, and detailed implications of the presence of the free neutrons, elsewhere.

The w-modes pose no real technical problems for our numerics and we can determine their frequencies and damping times to about 10 digit precision. The results indicate that the inclusion of a more accurate treatment of the matter content in the star does not influence the w-mode spectrum beyond the $\sim 10^{-8}$ level. This could have been anticipated since the w-modes depend almost entirely on the gravitational field of the background star. The background model is hardly affected at all by a more refined matter modelling, $e.g.$ in the crust region.

The plan of the paper is as follows. In section 2 we briefly review the formalism needed for a general relativistic treatment of the dynamics of solids coexisting with a (super)fluid. This is followed, in section 3, by an explanation of how the free neutrons affect the equations of motion in the linear perturbation regime. Our numerical results are then presented in section 4. We conclude the paper with section 5 and a discussion of the results and possible future developments. The numerical machinery used to solve the equations is summarised in an Appendix.

2 Formalism

We begin our analysis by introducing the basic framework used to describe the dynamics of a superfluid immersed in a solid lattice. The discussion is based on the theory developed by Carter & Samuelsson (CS) [20], to which we refer for details. We will, however, adapt our discussion to make maximal use of the formalism developed in a series of papers by Karlovini *et al.* dealing with pure solids [21, 22, 23, 24], hereafter referred to as Papers I-IV. We will also, as much as possible, adapt the notation to the recent review of relativistic fluids by Andersson & Comer [25], hereafter AC.

The theory is derived from a Lagrangian "master function" Λ which plays the dual role of providing the matter part of the total action and also giving the equation of state (EoS). For the purposes of the present study it is convenient to decompose the Lagrangian as (see CS)

$$\Lambda = \Lambda_{\mathrm{liq}} + \Lambda_{\mathrm{sol}} \tag{1}$$

where Λ_{liq} is a function of the particle fluxes n_{c}^a and n_{f}^a, while the elastic properties of the solid lattice are contained within Λ_{sol}. The functional Λ_{liq} is the contribution due to the liquid properties and corresponds to the full Lagrangian for a two-fluid system (see *e.g.* AC). We use the constituent indices 'c' and 'f' to distinguish the "confined" baryons in the lattice and the "free" neutrons, respectively. We assume a metric signature of the form $(-, +, +, +)$ and use early Latin letters, 'a', 'b', 'd', ... to denote abstract spacetime indices, see *e.g.* [26], omitting 'c' in order to avoid confusion. We will employ liberal index positioning to avoid unnecessary cluttering of the formulae. For instance, the total free neutron number density is given by

$$n_{\mathrm{f}}^2 = -n_{\mathrm{f}}^a n_a^{\mathrm{f}} = -g_{ab} n_{\mathrm{f}}^a n_{\mathrm{f}}^b \tag{2}$$

and similar for the confined baryon number density n_{c}.

The liquid contribution can be further split into a term that is independent of the relative velocity and a piece describing the effects of the relative current,

$$\Lambda_{\mathrm{liq}} = -\check{\rho} + \Lambda_{\mathrm{ent}} \tag{3}$$

Here we denote the velocity independent term by $\check{\rho}$ since it corresponds to the comoving unsheared energy density (which is uniquely defined only for zero relative velocity) as described, *e.g.* in Paper I. This quantity is assumed to describe the minimum energy for a given total baryon density $n = n_{\mathrm{c}} + n_{\mathrm{f}}$ and we will therefore consider it to be a function of n only. The remaining term arises because the state of matter may explicitly depend on the relative velocity between the different species of particles in a multi-fluid system. This leads to an effect known as entrainment. If only low relative velocities are considered (as will be the case here) it is sufficient to consider a slow motion approximation of the entrainment contribution. Since $\check{\rho}$ represents a minimum energy state for a given baryon density n it is clear that the leading order term in the entrainment term must be quadratic. The question is in what? There are at least two choices. In CS the expansion was carried out in terms of the relative flux, $n_\perp^a$, which, geometrically, can be defined locally as the projection of the neutron current onto the space orthogonal to the four velocity u^a of the solid (or vice versa),

$$n_\perp^a = h^a{}_b n_{\mathrm{f}}^b, \qquad h_{ab} = g_{ab} + u_a u_b \tag{4}$$

On the other hand, in micro-physical calculations it seems customary [27, 28] to use the relative velocity v^a (or some proxy of this) which is related to the neutron current through

$$n_{\mathrm{f}}^a = \gamma n_{\mathrm{f}}(u^a + v^a), \qquad u^a v_a = 0, \qquad \gamma = (1 - v^2)^{-1/2}, \qquad v^2 = v^a v_a \tag{5}$$

Combining (4) and (5) we see that

$$n_\perp^a = \gamma n_f v^a \quad \Rightarrow \quad n_\perp^2 = n_f^2 \frac{v^2}{1 - v^2} \approx n_f^2 v^2 (1 + v^2) \tag{6}$$

This analysis shows that the two prescriptions agree to $O(v^2)$. However, we also find that the equations of motion, which will depend on derivatives of the master function, may differ at $O(v^2)$ depending on the

choice of expansion and the level of truncation. In order to be consistent, one should therefore be careful to employ the same scheme as the microphysics calculation on which one bases the continuum model. This level of consistency is difficult to reach given our present (rather basic) level of understanding, but it sets the standard that future work should aspire to.

In this study we will consider axial linear perturbations of a static background model. The incompressible nature of this kind of motion together with the vanishing of the relative velocity in the background implies that we will not, even in principle, need to know the master function beyond $O(v^2)$. Thus, for the present purposes the problem mentioned above is a non-issue. For more general backgrounds and compressible motion (*e.g.* associated with acoustic oscillation modes) caution is advised.

Since the microphysics calculations we take our parameters from are performed as an expansion in v this is what we will use. Then, making the standard assumption that the liquid is intrinsically isotropic, we can write [27, 28]

$$\Lambda_{\text{ent}} = \frac{1}{n_{\text{c}}} m_{\text{c}}^{\text{f}} (n_{\text{cf}}^2 - n_{\text{c}} n_{\text{f}}) = n_{\text{f}} m_{\text{c}}^{\text{f}} (\gamma - 1) = \frac{1}{2} n_{\text{f}} m_{\text{c}}^{\text{f}} v^2 + O(v^4) \tag{7}$$

where $m_{\text{c}}^{\text{f}} = m_{\text{f}}^* - m$, m_{f}^* is the effective (dynamical) mass of the neutrons, m is the neutron mass (which we may take to be equal to the proton mass) and $n_{\text{cf}}^2 = -n_{\text{c}}^a n_a^{\text{f}}$.

The solid contribution is most easily described by an isotropic quasi-Hookean [29, 20] prescription. Then it is simply assumed that $\Lambda_{\text{sol}} = -\breve{\mu} s^2$ where $\breve{\mu} = \breve{\mu}(n_{\text{c}})$ is the shear modulus and s is a scalar measure of the state of strain. Although this description is likely to be adequate in any realistic situation we will nevertheless use the more elaborate description discussed in Paper I and only later, in the applications, specialise to isotropic quasi-Hookean solids. This will make the resulting equations formally valid for anisotropic solids without assuming the Hookean approximation. We will still use a minimal coupling ansatz in the sense that we (quite reasonably) assume that the solid contribution is independent of the free neutrons. By using the more general matter description, we have the advantage that we can adopt the results of Paper IV more or less directly.

When describing solids it is important to keep track of the reference state relative to which the strain is measured. For many simple solids it is possible to do this via a positive definite metric tensor field, the matter space metric k_{ab} (see Paper I). One may intuitively think of this tensor as encoding the (3-)geometry of the solid (as seen by the solid itself) in an unstrained state. The strain tensor is then given by

$$s_{ab} = \frac{1}{2}(h_{ab} - n_{\text{c}}^{-2/3} k_{ab}) \tag{8}$$

where h_{ab} is defined in (4). It is advantageous to work with an orthonormal eigenbasis e_μ^a of k_{ab} in which

$$k_{ab} = \sum_{\mu=1}^{3} n_\mu^2 e_\mu^a e_\mu^b \tag{9}$$

where we use Greek indices (with no implied sums) to enumerate the basis and n_μ are, loosely speaking, linear particle densities. In principle we could simply take the solidity contribution to the master function to be a function of the n_μ's. For practical purposes, however, it is convenient to instead work with the eigenvalues $\alpha_\mu = \frac{n_\mu}{n_{\text{c}}^{1/3}}$ of

$$\eta_{ab} = n_{\text{c}}^{-2/3} k_{ab} = \sum_{\mu=1}^{3} \alpha_\mu^2 e_\mu^a e_\mu^b \tag{10}$$

Since the determinant of the matter space metric is given by $\det(k^a{}_b) = n_{\text{c}}^{2/3}$ it is clear that

$$n_{\text{c}} = \Pi_{\mu=1}^{3} n_\mu \tag{11}$$

It follows that $\eta^a{}_b$ has a unit determinant so that this tensor only has two independent components. These components hold the key information of the material's response to non-compressional distortions and therefore encode the difference from a liquid. Within the eigenvalue formulation it is natural to prescribe the

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solid contribution as a function of the number density of confined baryons, n_c and the parameters α_μ,

$$\Lambda_{\text{sol}} = \Lambda_{\text{sol}}(n_c, \alpha_\mu) \tag{12}$$

making manifest the "minimal coupling" ansatz, *i.e.* that we assume that the solid's response to deformations is independent of the number density of free neutrons. In a quasi-Hookean approximation the solid's function of state may be separated in the form

$$\Lambda_{\text{sol}} = \check{\mu}(n_c)s^2(\alpha_\mu) \tag{13}$$

where $\check{\mu}$ is the shear modulus and s^2 is a scalar measure built from invariants of the strain tensor (see Paper I for a discussion).

To summarise, the above prescription leads to a Lagrangian that can be written

$$\Lambda = \Lambda_{\text{liq}}(n_c^a, n_f^a) + \Lambda_{\text{sol}}(n_c, \alpha_\mu) \tag{14}$$

where the standard isotropic two-fluid Lagrangian takes the form

$$\Lambda_{\text{liq}} = -\check{\rho} + \frac{1}{n_c}(m_f^* - m)(n_{cf}^2 - n_c n_f) \tag{15}$$

Practically speaking, our matter model is described by three functions of state; the minimum energy density $\check{\rho}$, the effective neutron mass m_f^* and the solid's function of state which, in the subsequent applications, will be encoded in the shear modulus $\check{\mu}$.

2.1 Equations of motion

The variational procedure described in CS, applied to the Lagrangian density discussed above, leads to a stress-energy tensor of the form

$$T^a{}_b = (\Lambda - n_f^d \mu_d^f - n_c^d \mu_d^c)\delta^a{}_b + n_f^a \mu_b^f + n_c^a \mu_b^c + \pi^a{}_b \tag{16}$$

Here

$$\mu_a^f = \frac{\partial \Lambda}{\partial n_f^a} = \frac{\partial \Lambda_{\text{liq}}}{\partial n_f^a}, \qquad \mu_a^c = \frac{\partial \Lambda}{\partial n_c^a} \tag{17}$$

are the momenta of the constituents and π_{ab} is the (trace-less) solid contribution as derived in Paper I. The stress energy tensor may be used as source in Einstein's equations. In addition, we have two equations of motion of the form

$$2n_f^a \nabla_{[a} \mu_{b]}^f = 0 \tag{18}$$

$$2n_c^a \nabla_{[a} \mu_{b]}^c + \nabla^a \pi_{ab} = 0 \tag{19}$$

where square brackets indicate anti-symmetrisation. In the following, we will use the fully variational description (CS). This means that the constituents are individually conserved, *i.e.* $\nabla_a n_c^a = \nabla_a n_f^a = 0$. The equations (18) and (19) (corresponding to the Euler equations) together imply the conservation of energy momentum $\nabla^a T_{ab} = 0$. This is, of course, also implied by the Einstein equations via the contracted Bianchi identities. Thus the various equations are not independent. This should be familiar from single fluid problem where it is well-known that one can opt to work with the Einstein equations alone. For multifluid problems more information is required. In the present context it is sufficient to consider a combination of the Einstein equations and one of the Euler equations. The simplest choice, since we only need to consider the complicated elastic terms once, is to let (18) do the job. As we will see, the incompressible nature of the axial modes together with the fact that the free neutrons are only forced via the entrainment then allows us to find the linearised solutions to the mode problem explicitly. We may also comment that the Cowling approximation, wherein the gravitational degrees of freedom are neglected, retain the property [6] that it is equivalent to consider the weak coupling limit of Einstein's equations together with (18). Alternatively, one

can decide to work only with (18) and (19). The latter strategy would lead to a problem which is trivially related to the corresponding one in Newtonian theory.

In order to obtain explicit equations we need to determine the momenta. Following the notation of AC we write formally

$$\mu_a^{\mathrm{f}} = \mathcal{B}^{\mathrm{f}} n_a^{\mathrm{f}} + \mathcal{A}^{\mathrm{cf}} n_a^{\mathrm{c}} \tag{20}$$

$$\mu_a^{\mathrm{c}} = \mathcal{B}^{\mathrm{c}} n_a^{\mathrm{c}} + \mathcal{A}^{\mathrm{cf}} n_a^{\mathrm{f}} \tag{21}$$

where

$$\mathcal{B}^{\mathrm{f}} := -2\frac{\partial \Lambda}{\partial n_{\mathrm{f}}^2} = 2\frac{\partial \check{\rho}}{\partial n_{\mathrm{f}}^2} + \frac{1}{n_{\mathrm{f}}} m_{\mathrm{c}}^{\mathrm{f}} + O(v^2) \approx \frac{1}{n_{\mathrm{f}}}(\check{\rho}' + m_{\mathrm{c}}^{\mathrm{f}}) \tag{22}$$

$$\mathcal{B}^{\mathrm{c}} := -2\frac{\partial \Lambda}{\partial n_{\mathrm{c}}^2} = 2\frac{\partial \check{\rho}}{\partial n_{\mathrm{c}}^2} + \frac{n_{\mathrm{f}}}{n_{\mathrm{c}}^2} m_{\mathrm{c}}^{\mathrm{f}} - 2\frac{\partial \Lambda_{sol}}{\partial n_{\mathrm{c}}^2} + O(v^2) \approx \frac{1}{n_{\mathrm{c}}}\left(\check{\rho}' + \frac{n_{\mathrm{f}}}{n_{\mathrm{c}}} m_{\mathrm{c}}^{\mathrm{f}}\right) - 2\frac{\partial \Lambda_{sol}}{\partial n_{\mathrm{c}}^2} \tag{23}$$

$$\mathcal{A}^{\mathrm{cf}} := -\frac{\partial \Lambda}{\partial n_{\mathrm{cf}}^2} = -\frac{1}{n_{\mathrm{c}}} m_{\mathrm{c}}^{\mathrm{f}} + O(v^2) \approx -\frac{1}{n_{\mathrm{c}}} m_{\mathrm{c}}^{\mathrm{f}} \tag{24}$$

Here we make use of the assumption that $\check{\rho}$ can be treated as a function of n only and denote its derivative with respect to n by a prime. The approximate expressions are valid up to $O(v)$ which is all we will need subsequently. We may note here that the generalised pressure (see, *e.g.* CS or AC) is

$$\Psi := \Lambda - n_{\mathrm{c}}^a \mu_a^{\mathrm{c}} - n_{\mathrm{f}}^a \mu_a^{\mathrm{f}} = n\check{\rho}' - \check{\rho} + \Lambda_{\mathrm{sol}} - n_{\mathrm{c}}\frac{\partial \Lambda_{sol}}{\partial n_{\mathrm{c}}} + O(v^2) \approx \check{p} + \Lambda_{\mathrm{sol}} - n_{\mathrm{c}}\frac{\partial \Lambda_{sol}}{\partial n_{\mathrm{c}}} \tag{25}$$

where $\check{p}$ denotes the usual (unsheared) pressure. Comparing to the result in Paper I we see that Ψ corresponds to the total isotropic pressure defined there. In an unstrained state we have

$$\check{\rho}' = \frac{\check{\rho} + \check{p}}{n} \approx \frac{\check{\rho}}{n} \approx m \tag{26}$$

The approximations are valid in the relatively tenuous neutron star crust where $\check{p} \ll \check{\rho}$ and $\check{\rho} \approx mn$. This simplification, which is accurate to within ~ 0.1 %, is obviously very useful since we replace a function with a known constant. Nevertheless, in our numerical code we use the full expression (which actually means that we do not need to assume that the neutrons and protons have the same mass).

In order to compare the present model to the purely elastic case considered in Paper I it is useful to express the energy momentum tensor in a frame adapted to the solid. Formally we then have

$$T_{ab} = \rho u_a u_b + 2u_{(a} Q_{b)} + P_{ab} \tag{27}$$

where ρ, Q_a and P_{ab} are, respectively, the total energy density, the momentum flow and the pressure tensor, all measured in a frame described by the solids four-velocity u^a. Putting the pieces together we find that

$$\rho = \rho_I + O(v^2) \tag{28}$$

$$Q_a = x_{\mathrm{f}}(\check{\rho} + \check{p})v_a + O(v^2) \tag{29}$$

$$P_{ab} = \Psi h_{ab} + \pi_{ab} + O(v^2) = P_{ab}^I + O(v^2) \tag{30}$$

where $x_{\mathrm{f}} = n_{\mathrm{f}}/n$ is the free neutron fraction and we use the label I to denote the corresponding quantity in Paper I. Thus, to order v the free neutrons lead to the presence of a non-zero momentum flow Q_a. Obviously, a static background is unaffected by this.

3 Perturbations

Let us now consider the equations governing axial perturbations around a static background. We base our derivation of these equations on the general framework developed by Karlovini [30] and make heavy use of the results in Paper IV to which we refer for details.

The starting point of the analysis is a decomposition of the metric in the form

$$g_{ab} = \perp_{ab} + F^{-1}\eta_a\eta_b, \qquad F = \eta^a\eta_a = (r\sin\theta)^2, \qquad \eta^a \perp_{ab} = 0 \tag{31}$$

where η^a is the axial Killing vector. Karlovini [30] showed that the axial part of the metric perturbations (denoted here by γ_{ab}) can be entirely expressed in terms of $\delta\eta_a$ so that δF and $\delta\perp_{ab}$ can both be set to zero[1]. The perturbed Einstein equations then take the simple form

$$\nabla_b(FQ^{ab}) = \kappa J^a \tag{32}$$
$$\nabla_a J^a = 0 \tag{33}$$

where, loosely speaking, $Q_{ab} = 2\nabla_{[a}F^{-1}\delta\eta_{b]}$ represent the geometric perturbations and

$$J^a = 2\delta(\perp^{ab} \eta^c T_{bc}) \tag{34}$$

encode the matter motion. Hence, it is natural to refer to J^a as the matter current. These equations were discussed in detail for the case of a single solid in Paper IV. Here we focus on the changes needed to extend those equations to the case where the solid is coexisting with a superfluid. As noted above, the superfluid degrees of freedom manifest themselves by i) the additional momentum flow Q^a in the energy momentum tensor, and ii) the need to consider the equation of motion (18). In order to work out the explicit perturbation equations we found it useful to make use of the eigenvector formulation introduced in Paper I where the principal directions of the solid provide an orthonormal tetrad $\{u^a, e^a_\mu\}$, where u^a is the four-velocity of the solid, $\mu \in \{1, 2, 3\}$ and we use Greek indices to enumerate the spatial basis vectors. The general expressions for the perturbed tetrad were given in Paper IV in terms of the perturbed metrics $\gamma_{ab} = \delta g_{ab}$ and δk_{ab} which in the axial case considered here can be written;

$$\gamma_{ab} = 2F^{-1}\eta_{(a}\delta\eta_{b)}, \qquad \delta k_{ab} = 2n_3^2\eta_{(a}\nabla_{b)}\delta\tilde{\phi} \tag{35}$$

The quantity $\tilde{\phi}$ is the (inverse) mapping of the azimuthal coordinate on matter space (see Paper IV) and we use parentheses to denote symmetrisation. In general, the inclusion of the free neutrons adds four scalar degrees of freedom. We take these to be represented by the free neutron density $n_{\rm f}$ and the relative velocity v^a (which, due to the constraint $u^a v_a = 0$ has three degrees of freedom). For the axial case it is straightforward to show that we must have (for a static background)

$$\delta n_{\rm c} = \delta n_{\rm f} = \delta n_{\rm cf} = 0 \tag{36}$$

It follows that

$$\delta\mathcal{B}^{\rm c} = \delta\mathcal{B}^{\rm f} = \delta\mathcal{A}^{\rm cf} = 0 \tag{37}$$

so that the perturbed momenta are given by

$$\delta\mu^{\rm c}_a = \mathcal{B}^{\rm c}\delta n^{\rm c}_a + \mathcal{A}^{\rm cf}\delta n^{\rm f}_a \tag{38}$$
$$\delta\mu^{\rm f}_a = \mathcal{B}^{\rm f}\delta n^{\rm f}_a + \mathcal{A}^{\rm cf}\delta n^{\rm c}_a \tag{39}$$

For the perturbations we shall consider the relative velocity to be aligned with the axial Killing vector. Therefore we may write it in the form

$$\delta v^a = v^a = v\eta^a \tag{40}$$

where we have dropped the perturbation symbol δ since no confusion can arise. This leads to a simple expression for the perturbed neutron momentum,

$$\delta\mu^{\rm f}_a = [(\check{\rho}' + m^f_c)v - \check{\rho}'u^b K_b]\eta_a \tag{41}$$

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[1]In order to avoid introducing yet another μ we have chosen to express the metric perturbations in terms of η_a rather than $\mu_a = F^{-1}\eta_a$ used in [30] and Paper IV. Note that $\delta\eta_a = F\delta\mu_a \neq g_{ab}\delta\eta^a = 0$ where the latter equality is due to a partial gauge fixing (which is just for convenience since the present formalism is gauge invariant).

where $K_a = \nabla_a \delta\tilde{\phi} - F^{-1}\delta\eta_a$ is a gauge invariant one-form. As discussed in paper IV, it encodes the complete nature of the perturbations of the solid. On a static background, and only considering the non-stationary (oscillatory) perturbations of equation (18) we find the remarkably simple result

$$\delta\mu_a^{\mathrm{f}} = 0 \tag{42}$$

This implies, via (41), that we have an algebraic solution for the perturbed relative velocity

$$v = \frac{\check{\rho}'}{\check{\rho}' + m_c^f} u^a K_a \approx \frac{m}{m_{\mathrm{f}}^*} u^a K_a \tag{43}$$

Note also that this shows that the vorticity remains zero to the first order accuracy considered here. Thus, no vortices are created or destroyed dynamically to this order. This simple result may seem surprising at first, but it is a direct consequence of the incompressible nature of axial perturbations. The variational principle we use is based on varying the flowlines of the particles. Incompressibility implies that the total number of flowlines in a 3-volume is conserved and hence restricts the variations further to the extent that only a single displacement vector will be needed to describe the motion of the full system.

Armed with the solution to the equation of motion for the free neutrons we are in a position to evaluate the effect the free neutrons have on the perturbed energy momentum tensor. As already pointed out, the only difference from the purely elastic case is the presence of the momentum flow Q_a. Perturbing (29) and inserting the result in (34) we readily find

$$J^a = 2(\rho + p_t)F\tilde{S}^{ab}K_b \tag{44}$$

where ρ and p_t are the total energy density and the tangential pressure, respectively, and the difference induced by the superfluid is the modification of the shear wave "metric":

$$\tilde{S}^{ab} = S^{ab} + x_{\mathrm{f}}\frac{\check{\rho}'}{\check{\rho}' + m_c^f}\frac{\check{\rho} + \check{p}}{\rho + p_t}u^a u^b \tag{45}$$

Here

$$S^{ab} = -u^a u^b + v_{r\perp}^2 e_1{}^a e_1{}^b + v_{t\perp t}^2 e_2{}^a e_2{}^b \tag{46}$$

is the corresponding metric for purely elastic matter expressed in terms of the eigenvector basis $e_\mu{}^a$ of the solid lattice. Meanwhile, $v_{r\perp}$ and $v_{t\perp t}$ are the shear wave velocities in the radial and tangential directions, respectively (see Paper I). Hence, the effect of the neutrons is <u>entirely</u> contained in the factor

$$\chi = 1 - x_{\mathrm{f}}\frac{\check{\rho}'}{\check{\rho}' + m_c^f}\frac{\check{\rho} + \check{p}}{\rho + p_t} \tag{47}$$

From Paper I we know that the change of the pressure and density due to anisotropy is of the order $\sim \check{\mu}s^2$. In the bulk of the crust we have the ordering $\check{\mu} \ll \check{p} \ll \check{\rho}$ and in addition $s^2 \lesssim 0.1$ [31]. Hence we expect that the approximations $\rho \approx \check{\rho}$ and $p_t \approx \check{p}$ will always hold in realistic neutron star crusts. They will, of course, be exactly true on an unstrained background. We therefore note that since $\check{\rho}' \approx m$ we have (to $\sim 0.1\%$ precision)

$$\chi \approx 1 - x_{\mathrm{f}}\frac{m}{m_{\mathrm{f}}^*} \tag{48}$$

in perfect agreement with result from a recent analysis of incompressible plane waves in the corresponding Newtonian problem [17]. Thus, to "upgrade" the equations in Paper IV we only need to insert the factor χ appropriately. The general equations for a static background and $l \geq 2$ thus become

$$-\dot{W}_t + W_r' - \frac{r'}{r}W_r - e^{2\nu}\frac{L}{r^2}\psi = 0 \tag{49}$$

$$-\dot{W}_r + W_t' + \frac{r'}{r}W_t + e^{2\nu}\frac{\mathcal{E}v_{t\perp t}^2}{r^2}\varphi = 0 \tag{50}$$

$$-Lr\dot{\psi} + \mathcal{E}v_{r\perp}^2 r^2 (r^{-1}\varphi)' + (\mathcal{E}v_{r\perp}^2 + L)rW_t = 0 \tag{51}$$

$$-\chi\mathcal{E}r\dot{\varphi} + L(r\psi)' - (\chi\mathcal{E} + L)rW_r = 0 \tag{52}$$

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cf. equations (98)-(101) in Paper IV. Here r is the radial measure in Schwarzschild coordinates in which the line element takes the form

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + e^{2\lambda}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{53}$$

We are also using $L = (l-1)(l+2)$,

$$\mathcal{E} = 2\kappa r^2(\rho + p_t) \tag{54}$$

and dots and primes refer to derivatives with respect to time and the Regge-Wheeler radial coordinate r_* which is given by

$$\mathrm{d}r_* = e^{\lambda-\nu}\mathrm{d}r \tag{55}$$

The total energy density ρ and the tangential pressure p_t (see Paper I) have been retained to keep the formulae valid for anisotropic backgrounds. We will later set them to the isotropic values $\check{\rho}$ and $\check{p}$, respectively. For an explanation of the dependent variables, see Paper IV. We see that only the last equation is changed by the presence of the free neutrons.

For completeness we also give the modified wave equations (of which only the second, describing the shear waves, is modified)

$$-\ddot{\mathcal{W}}_t + \mathcal{W}_t'' + e^{2\nu}\left(\frac{6m}{r^3} - \frac{1}{2}\kappa(\rho - p_r) - \frac{\mathcal{E}v_{r\perp}^2}{r^2} - \frac{l(l+1)}{r^2}\right)\mathcal{W}_t$$
$$= r\left[\frac{e^{2\nu}\mathcal{E}(v_{r\perp}^2 - v_{t\perp t}^2)}{r^3}\varphi\right]' - \left[\frac{e^{2\nu}\mathcal{E}v_{r\perp}^2}{r^2}\right]'\varphi \tag{56}$$

$$-\chi\ddot{\varphi} + \frac{(\mathcal{E}v_{r\perp}^2\varphi')'}{\mathcal{E}} - \left[\frac{(\mathcal{E}v_{r\perp}^2 r')'}{\mathcal{E}r} + e^{2\nu}\frac{v_{t\perp t}^2(\chi\mathcal{E} + L)}{r^2}\right]\varphi = \frac{[\mathcal{E}(\chi - v_{r\perp}^2)r\mathcal{W}_t]'}{\mathcal{E}r} - \frac{\chi\mathcal{E}'}{\mathcal{E}}\mathcal{W}_t \tag{57}$$

The boundary conditions, namely that $\mathcal{W}_t$, $\mathcal{W}_r$ and ψ are everywhere continuous, do not change. Thus, we now have all the information needed to solve the axial oscillation problem including the free neutrons.

4 Results

The basic strategy for solving the perturbation equations (49) – (52), in the case where the free neutrons are ignored, ($\chi = 1$) has already been outlined in Paper IV. Given that the free neutrons do not change the problem formally, the same strategy can be used in the more general setting discussed here. Nevertheless, it is worthwhile providing some detail on the methods we have used to to solve the equations. Interested readers can find a discussion of the relevant points in the Appendix.

We build our background neutron star models using realistic tabulated equations of state, see table 1. The two older equation of state tables "A" [32] and "B" [33] were taken from the distribution of the `rns` code, see *e.g.* [36], whereas "APR" [34] was provided by G. Ravenhall. In the crust the equation of state is taken from Douchin & Haensel [37] (DH)[2] and we use the shear modulus for a Coulomb lattice as calculated by Ogata & Ichimaru [39][3]. The effective mass needed for the entrainment was taken from Chamel [41]. We emphasise that, since his calculations were performed for a different EoS than the ones we use, our model for the effective mass is inconsistent. In addition, we are only aware of data for four particular densities. Since we need data for all crust densities, we use an analytic expression that approximately passes through the available data points, see [17]. This is, obviously, not satisfactory but it is the best that we can do at the present time. Further work on the properties of the crust "beyond" the EoS (minimum energy state) should be encouraged. Even though our model is somewhat *ad hoc*, it should provide insights into the basic dynamics of the problem.

As a first test of the numerical code we determined the w-modes for a number of cases:

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[2]In fact, in the outer crust we employ the results described in [38] using more recent measurements for the binding energy of nuclei, but the end result is practically indistinguishable from DH.

[3]The effects of using the very recent results of Horowitz & Hughto [40], obtained from molecular dynamics simulations, will be examined in a forthcoming paper.

Model	EoS	$\check{\rho}_0$ (10^{15} g/cm^3)	$M(M_\odot)$	R (km)	$M_{\text{core}}(M_\odot)$	R_{core} (km)	β
A1(a)	A	1.259	1.04959	9.89188	-	-	0.156751
A1(b)	A + DH	1.259	1.04806	9.90598	1.03690	8.99246	0.156300
A2(a)	A	4.110	1.65241	8.37048	-	-	0.291632
A2(b)	A + DH	4.110	1.65019	8.37450	1.64746	8.10619	0.291100
B1(a)	B	1.995	0.97068	8.76793	-	-	0.163549
B1(b)	B + DH	1.995	0.97045	8.78057	0.96302	8.01716	0.163274
B2(a)	B	5.910	1.41131	7.07165	-	-	0.294828
B2(b)	B + DH	5.910	1.41003	7.07679	1.40843	6.85589	0.294348
APR1(a)	APR	0.750	0.91989	11.5963	-	-	0.117189
APR1(b)	APR + DH	0.750	0.92026	11.7203	0.89627	10.1619	0.115995
APR2(a)	APR	2.750	2.19428	10.0059	-	-	0.323969
APR2(b)	APR + DH	2.750	2.19436	10.0238	2.19071	9.77753	0.323404

Table 1: Background models used in our study. We consider three equations of state, labelled A [32], B [33] and APR [34]. The reason for this particular choice is simply that they were previously studied by BBF [35]. This allows us to compare the results for the w-modes directly. For each equation of state we consider the same two central densities as BBF. For each of these models we also substitute the equation of state by DH in the crust. As the data in the table shows, this leads to slightly different total masses and radii. For each model we provide the central density ρ_0, total mass M, total radius R and compactness $\beta = M/R$. For the models where the crust-core transition pressure is known (*i.e.* for the DH crust) we also give the mass M_{core} and radius R_{core} of the core.

(1a) using the fluid equations with the available tables, as described above, all the way to the surface,
(1b) as (1a) but substituting using the EoS of Douchin & Haensel in the crust region
(2) artificially setting the shear modulus to zero,
(3) using the full equations but artificially setting $\chi = 1$, and finally,
(4) using the full equations including the model for the superfluid neutrons.

Case (1a) can be directly compared to the results of Benhar *et al.* [35] (BBF). We compare a selection of typical results in table 2. This comparison shows that the results are generally in good agreement. The small differences could be due to slightly different tabulations of the EoS or different interpolation schemes (we use logarithmic interpolation). We also expect our treatment of the surface of the background model to be more accurate. To rule out the possibility that the difference is due to a bug in our code we also calculated w-modes for a polytropic equation of state and compared to the results of Andersson & Kokkotas [42]. In this case (where the ambiguity in the treatment of the EoS is eliminated) we find perfect agreement. A third test of the code is provided by comparing cases (1b) and (2). The agreement is excellent also in this case.

In order to investigate what effect the elastic solid and the superfluid neutrons have on the w-modes it is relevant to compare cases (2)–(4). Since the w-modes are of gravitational origin and the matter motion is limited to the relatively tenuous crust we expect a very small effect. The numerical results confirm this expectation. We find that the relative influence of the crust and the superfluid is less than $\sim 10^{-8}$ (see table 3 for some typical results). This difference is roughly at the same level as the expected accuracy of our code. Thus, for all practical purposes, the axial w-modes are unaffected by the crust properties.

Turning to the torsional t-modes, where one would expect the superfluid to have a significant effect, we compare cases (2) and (3) to our previous results [6], that were obtained within the Cowling approximation.

Model	$f_{(a)}$ (kHz)	$f_{(b)}$ (kHz)	f_{BBF} (kHz)	$\tau_{(a)}$ (μs)	$\tau_{(b)}$ (μs)	τ_{BBF} (μs)
A1	9.781128	9.785983	9.76	21.574378	21.540601	21.6
A2	9.107043	9.119139	9.11	72.028558	71.577816	72.4
B1	11.24895	11.25115	11.2	20.003988	19.999529	20.2
B2	10.65463	10.66564	10.6	70.601159	70.188532	71.7
APR1	8.865023	8.865084	8.82	19.195766	19.203608	19.4
APR2	6.725440	6.725176	6.69	158.52087	158.53555	165.3

Table 2: The data in this table compares the w-modes that we have determined to the results of BBF. We give the frequencies and damping times for the lowest $l = 2$ w-modes for completely fluid stars. The subscripts refer to the cases discussed in the text, *i.e.* (a) uses the tables discussed in the main text and the fluid equations, (b) substitutes the DH EoS in the crust and use the solid equations with $\check{\mu}$ set to zero. Finally, BBF refers to the results in [35].

Model	f (kHz)	τ (ms)
Fluid	9.11913825	0.0715778249
Solid	9.11913834	0.0715778266
Solid + SF	9.11913833	0.0715778264

Table 3: The data in this table illustrates the effects of the matter model. We give the frequency and damping time of the lowest $l = 2$ w-mode for the background model A2(b) (see table 1) for different matter models. The three models are: a completely fluid star ($\check{\mu} = 0$, 'Fluid'), a star with a solid crust, but neglecting the free neutrons ('Solid') and the complete model including the neutrons ('Solid +SF'). The results show that the effects due to the more detailed treatment of the crust physics are tiny. The results have converged (after increasing the resolution) to the stated precision, but one should be aware of the fact that the roundoff error in the calculation is, as discussed in the main text, of the order of $\sim 10^{-10}$.

Since the estimated (via the quadrupole formula) gravitational-wave damping time[4] for the t-modes is about $\tau \sim 10^4$ years [19] we expect the real part of the frequency to be well approximated by the Cowling results and the imaginary part to be very small. Indeed, a typical $l = 2$ fundamental torsional mode has $f \sim 30$ Hz so that

$$\Re(\omega) = 2\pi f \sim 1.9 \times 10^2 \text{ s}^{-1} \tag{58}$$

In contrast, the imaginary part is

$$\Im(\omega) = \frac{1}{\tau} \sim 0.8 \times 10^{-12} \text{ s}^{-1} \tag{59}$$

some fourteen orders of magnitude smaller than the real part. Since our numerical code uses an adaptive step Runge-Kutta integrator we can control the accuracy in each step. However, using a small error tolerance per step induces an uncontrollable truncation error (we use double precision so that the truncation error is $\sim 10^{-15}$) which, if turning down the tolerance on the error in each step, requires more steps to complete the integration. A simple analysis, using the *de facto* used steps together with the prescribed tolerance, suggests that our code cannot reach an accuracy beyond about 1 part in 10^{10}. Thus, we should not expect to be able to determine the the damping time of the t-modes directly. The numerical calculations confirm this

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[4]Note that the estimates in [19] concern the damping time for the energy, while we need the damping time associated with the amplitude. However, given the order of magnitude nature of our discussion of this point we do not bother with the different factors of 2 that relate these quantities.

Model	$_2f_0$ (Hz)	$_2f_1$ (Hz)	$_2f_2$ (Hz)
A1(b)	31.097 (31.069)	881.15 (881.15)	1449.2 (1449.4)
A1(b) + SF	33.897 (33.872)	957.93 (957.92)	1537.7 (1537.8)
A2(b)	27.169 (27.160)	1794.2 (1794.2)	2962.9 (2963.3)
A2(b) + SF	29.672 (29.665)	1950.3 (1950.2)	3164.7 (3165.1)
B1(b)	34.549 (34.525)	1031.5 (1031.5)	1697.3 (1697.5)
B1(b) + SF	37.666 (37.645)	1121.4 (1121.4)	1802.1 (1802.2)
B2(b)	31.874 (31.867)	2144.8 (2144.8)	3542.1 (3542.5)
B2(b) + SF	34.812 (34.806)	2331.4 (2331.4)	3783.6 (3784.0)
APR1(b)	28.887 (28.841)	584.00 (584.00)	955.60 (955.66)
APR1(b) + SF	31.449 (31.408)	634.90 (634.89)	1008.0 (1008.1)
APR2(b)	20.739 (20.731)	1649.4 (1649.4)	2724.7 (2725.0)
APR2(b) + SF	22.655 (22.647)	1792.9 (1792.9)	2912.1 (2912.5)

Table 4: This table provides a sample of results for the first few quadrupole ($l = 2$) t-modes. The background models are explained in table 1 and the extra label "+ SF" indicate that we have taken the free neutrons into account. The results using the Cowling approximation are shown within parenthesis for comparison. Since the code we use for the Cowling approximation is optimised for speed rather than accuracy it can only deliver 5-6 digit precision which is why we only quote this accuracy here. The calculations reported in this work is much higher, but this is most likely irrelevant for astrophysical applications. We note that the Cowling approximation in the cases tested is accurate to better than about 0.01 %. We also note that the case where the free neutrons are taken into account typically gives $\sim$ 10 % higher frequencies, but we stress that this is model dependent and the effects can be much larger (or smaller).

expectation. When we feed the output of the integrator into the Müller root-solver we find that we cannot determine the imaginary part of the mode frequency. The real part, on the other hand, converges nicely to more than ten digit precision (while the imaginary part oscillates around zero). For this reason we are confident that we determine the oscillation frequency accurately. Typical results are provided in table 4. In the table, we compare the results obtained using the Cowling approximation (the case where the superfluid is neglected was discussed in [6] and the analysis of the problem including the superfluid is currently being finalised).

5 Discussion

In this paper we have considered the global axial quasi-normal modes of relativistic stars with a (possibly) superfluid core and a solid crust penetrated by a superfluid component. Our results provide the first detailed analysis of this problem in general relativity, and represent a key step towards the modelling of realistic neutron star dynamics. In fact, apart from the linear approximation common to all mode studies and the omission of the magnetic field, our treatment does not impose any significant approximations[5]. The discussion does, however, highlight the need for improved microphysics models. While we have made an effort to use models that are as "realistic" as possible, it is clear that the input parameters that we have used are somewhat inconsistent. This is entirely due to the lack of complete data from microphysics studies. Future

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[5]The treatment of the solid component assumes a conformally deforming solid, see Paper I, but as this class incorporates, *e.g.* cubic symmetric lattices and isotropic solids we do not think that this constraint imposes any important restrictions at the present time.

tabulated equations of state need to provide, in particular, the superfluid entrainment parameters (both in the crust and in the core). Once such data becomes available, it will be straightforward to incorporate it in our computational framework. The issue concerning the neutron star magnetic field is more challenging. This problem will require serious thought, especially if we want to account for the presence of superconducting protons in the core.

Specialising to an isotropic solid we determined both the axial gravitational w-modes and the elastic torsional t-modes. In the case of the w-modes we calculated both the frequency and the damping time with high precision. The obtained results confirm the expectation that these modes are very weakly influenced by the presence of a solid/superfluid component. We also confirmed that the frequencies of the t-modes are only weakly affected by the coupling to the gravitational degrees of freedom. We were, however, unable to directly determine the damping time for these modes. This is likely to be completely irrelevant from an astrophysical point of view, since other mechanisms will dominate the damping of these modes. Of course, from an academic point of view the situation is not entirely satisfactory. As a matter of principle, one would like to have a direct determination of the damping times. Of course, a direct application of the quadrupole formula should give reliable results. A possible future option would be to take advantage of the enormous difference between the real and imaginary parts of the quantities that appear in the perturbation equations and perform a "second perturbation". One can easily check that this leads to a set of decoupled equations for the real parts and a set of equations for the imaginary parts sourced by the real parts. We have not yet tried to implement such a scheme.

As far as any magnetar seismology analysis is concerned [6], the main lesson to learn from our analysis is that one does not need an accurate treatment of the gravitational degrees of freedom. However, the presence of the superfluid component in the crust is important. We are currently investigating this problem in more detail within the relativistic Cowling approximation, and hope to be able to report on the results in the near future. Having developed the framework for the axial perturbations for relativistic neutron star models with elastic and superfluid components, we also need to consider the (generally more complex) problem of polar perturbations. Developments in this direction are also under way.

Acknowledgements

A   Appendix: Numerical formalism

In this appendix we discuss the methods we used to solve the axial perturbation equations (49) – (52). In paper IV the problem was analysed for the case $\chi = 1$, *i.e.* when the free neutrons are ignored. Given that there is no real formal difference between the two systems that need to be solved, we can use the same strategy also for this more general problem.

In the interior of the star we use a standard Runge-Kutta ODE solver from the GSL libraries [43]. The exterior perturbations are determined by a pseudo-spectral method (see [44] and below).

A.1   Background solution

A static, spherically symmetric, unstrained solution does not depend on the elastic/superfluid nature of the matter content since both the strain and entrainment contributions vanish. Thus, the background solution is, in this case, identical to that of a perfect fluid. It is well known that the use of a radial coordinate as independent variable, as in the standard TOV equations, leads to numerical difficulties near the surface. One problem is the steep gradient of the fluid variables near the surface and another is that the surface itself is determined by the condition that the radial pressure vanishes. That is, it is determined by one of the

dependent variables. Lindblom [45] suggested that a better strategy, as far as the background is concerned, is to define the two dependent variables

$$u = r^2, \qquad v = \frac{m}{r} \tag{60}$$

and use the relativistic enthalpy h defined by

$$dh = \frac{d\check{p}}{\check{\rho} + \check{p}} \tag{61}$$

as the independent variable. Then the equations of structure take the form

$$\frac{du}{dh} = -\frac{4u(1-2v)}{\kappa \check{p} u + 2v} =: u' \tag{62}$$

$$\frac{dv}{dh} = -(1-2v)\frac{\kappa \check{\rho} u - 2v}{\kappa \check{p} u + 2v} = \frac{u'}{4u}(\kappa \check{\rho} u - 2v) \tag{63}$$

In order to ensure a regular centre the integration is started at finite radius using the expansion

$$u = \frac{12(h_0 - h)}{\kappa(\check{\rho}_0 + 3\check{p}_0)} + \dots \tag{64}$$

$$v = \frac{2\check{\rho}_0(h_0 - h)}{\check{\rho}_0 + 3\check{p}_0} + \dots \tag{65}$$

where a subscript '0' will be used throughout our discussion to denote evaluation at the centre.

A.2   Perturbations

As we are interested in quasi-normal modes we make the standard assumption that the time dependence is given by $\exp(i\omega t)$ where ω is a complex constant whose real and imaginary parts represent the angular frequency and the damping timescale, respectively. Inside the star we solve the perturbation equations together with the background equations.

A.2.1   The vacuum region

In the surrounding vacuum region the axial perturbation equations reduce to the standard Regge-Wheeler equation. For quasi-normal modes we require that the solutions are outgoing waves at null infinity. This problem was discussed in detail by Samuelsson, Andersson and Maniopoulou [44] and we apply their code as it is. Then, given the mass M, radius R and angular frequency ω we obtain the surface value g_s of a certain phase function g,

$$g_s = g_{|r=R} = \left(\frac{1}{\psi}\frac{d\psi}{dr}\right)\bigg|_{r=R} + \frac{i\omega R}{R - 2M} \tag{66}$$

A.2.2   The fluid core

Following Paper IV we adapt the independent variables to the centre of the star by defining

$$\mathcal{X}_1 = r^{-l-1}\mathcal{W}_t \tag{67}$$

$$\mathcal{X}_2 = -i\omega e^{h_0 - \nu_s} r^{-l}\mathcal{W}_r \tag{68}$$

$$\mathcal{X}_3 = -i\omega e^{h_0 - h} r^{-l-1}\psi \tag{69}$$

$$\mathcal{X}_4 = \omega^2 e^{2h_0 - h - \nu_s} r^{-l}\varphi \tag{70}$$

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where $\nu_s = \frac{1}{2}\ln\left(1 - \frac{2M}{R}\right)$ denotes the surface value of ν. In the fluid region the equations then reduce to

$$\frac{d\mathcal{X}_1}{dh} = -\frac{u'}{2u}\left[(l+2)\mathcal{X}_1 + \frac{e^{h-h_0}}{\sqrt{1-2v}}\mathcal{X}_2\right] \tag{71}$$

$$\frac{d\mathcal{X}_2}{dh} = -\frac{u'}{2u}\left\{\frac{e^{h-h_0}}{\sqrt{1-2v}}\left[(l+2)(l-1)e^{2(h_0-h)} - \omega^2 e^{2(h_0-\nu_s)}u\right]\mathcal{X}_1 + (l-1)\mathcal{X}_2\right\} \tag{72}$$

and the constraints

$$\mathcal{X}_3 = -e^{h_0-h}\mathcal{X}_1, \qquad \mathcal{X}_4 = -e^{h_0-h}\mathcal{X}_2 \tag{73}$$

We may note here that the perturbations in the core do not depend on the multifluid nature of the medium (*i.e.* χ does not appear). This is due to the fact that, like an ordinary perfect fluid, this type of matter cannot sustain axial oscillations, but only stationary currents. Hence, the only oscillations that remain are those associated with the gravitational degrees of freedom (the w-modes). This means that, one cannot use the axial w-modes to distinguish between non-rotating single- and multi-fluid stars. If the star is rotating, however, it is known that the "axial-led" inertial modes will depend on the superfluid nature of matter [46].

In order to make sure that we have a regular solution near the origin we expand the solution according to

$$\mathcal{X}_1 = \hat{\mathcal{X}}_1\left(1 - \frac{6e^{2(h_0-\nu_s)}\omega^2 + \kappa(l+2)[3\check{p}_0 - (2l-1)\check{\rho}_0]}{(2l+3)\kappa(\check{\rho}_0 + 3\check{p}_0)}(h_0 - h) + \ldots\right) \tag{74}$$

$$\mathcal{X}_2 = \hat{\mathcal{X}}_1\left(-(l+2) + \frac{6(l+4)e^{2(h_0-\nu_s)}\omega^2 - (l+2)(l-1)\kappa[3\check{p}_0 + (2l+7)\check{\rho}_0]}{(2l+3)\kappa(\check{\rho}_0 + 3p_0)}(h_0 - h) + \ldots\right) \tag{75}$$

where $\hat{\mathcal{X}}_1$ represent the arbitrary scaling of the solutions and one should keep in mind that we are allowed to rescale the solutions at our convenience. We check that the solution does not depend on the (small) value chosen for $h_0 - h$.

A.2.3 The crust

In the crust we have chosen to integrate a slightly different set of equations with dependent variables given by

$$\mathcal{Y}_1 = r\mathcal{W}_t \tag{76}$$

$$\mathcal{Y}_2 = \frac{1}{i\omega r}\mathcal{W}_r \tag{77}$$

$$\mathcal{Y}_3 = r(\mathcal{W}_t - i\omega\psi) \tag{78}$$

$$\mathcal{Y}_4 = \frac{1}{r}\varphi + \frac{1}{i\omega r}\mathcal{W}_r \tag{79}$$

For these variables $\mathcal{Y}_{3-4}$ are constrained to vanish in a fluid and $\mathcal{Y}_3 = 0$ also in vacuum. Moreover, $\mathcal{Y}_{1-3}$ are everywhere continuous. Since we lack concrete information about any anisotropy in the solid we specialise the equations to the isotropic case where

$$v_{r\perp}^2 = v_{t\perp t}^2 = \frac{\check{\mu}}{\check{\rho} + \check{p}}, \qquad \rho = \check{\rho}, \qquad p_t = p_r = \check{p} \tag{80}$$

The set of equations we solve then becomes

$$\mathcal{Y}_{1,h} = -Pu\left[\omega^2\mathcal{Y}_2 - 2\kappa e^{2(\nu_s-h)}\check{\mu}(\mathcal{Y}_2 - \mathcal{Y}_4)\right] \tag{81}$$

$$\mathcal{Y}_{2,h} = \frac{P}{u^2\omega^2}\left[\omega^2 u\mathcal{Y}_1 - Le^{2(\nu_s-h)}(\mathcal{Y}_1 - \mathcal{Y}_3)\right] \tag{82}$$

$$\mathcal{Y}_{3,h} = \frac{2\kappa Pu}{L}\left[\chi\omega^2 u(\check{\rho} + \check{p})\mathcal{Y}_4 + Le^{2(\nu_s-h)}\check{\mu}(\mathcal{Y}_2 - \mathcal{Y}_4)\right] \tag{83}$$

$$\mathcal{Y}_{4,h} = -\frac{LP}{2\kappa u^2\omega^2\check{\mu}}\left[\omega^2\mathcal{Y}_3 + 2\kappa e^{2(\nu_s-h)}\check{\mu}(\mathcal{Y}_1 - \mathcal{Y}_3)\right] \tag{84}$$

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where $P = u'e^{\nu_s - h}/2\sqrt{u(1-2v)}$.

A.2.4 Boundary conditions

As outlined above, the boundary conditions both at the centre of the star and at infinity are already taken care of in our scheme. It thus remains to ensure that the proper jump conditions are satisfied at the top and bottom of the crust. At these interfaces we must ensure that $\mathcal{Y}_{1-3}$ are continuous. As discussed in Paper IV, we have no local conditions on the fourth variable. However, a generic choice of $\mathcal{Y}_4$ at the inner (say) boundary will not admit a continuous solution at the outer boundary. For this reason we solve for two linearly independent solutions in order to find the unique[6] linear combination that satisfies all boundary conditions.

There is one difficulty with this approach, however. The first term in equation (84) is proportional to $\mathcal{Y}_3/\breve{\mu}$. Near the surface of the star the boundary conditions require that $\mathcal{Y}_3 \rightarrow 0$ which balances the smallness of the shear modulus $\breve{\mu}$ in that region. Unfortunately, neither of the two outgoing solutions that we compute would typically satisfy the boundary conditions at the surface (only a specific linear combination will). Hence, this term will typically become very large close to the surface which makes it difficult to find an accurate solution. To remedy this situation we integrate the eigenfunctions in the crust both from the top and the bottom to some intermediate matching point. We check that the choice of matching point does not affect the solutions.

The explicit matching conditions are as follows. At the fluid-solid interface we have

$$\mathcal{Y}_1 = r^{(l+2)}\mathcal{X}_1 \tag{85}$$

$$\mathcal{Y}_2 = e^{\nu_c}r^{(l-1)}\omega^{-2}\mathcal{X}_2 \tag{86}$$

$$\mathcal{Y}_3 = 0 \tag{87}$$

$$\mathcal{Y}_4 = \text{free} \tag{88}$$

where the $\mathcal{X}_i$'s are given by the fluid solution. We define the two linearly independent solutions with the interface values

$$\mathcal{Y}_i^{(1)} = [r^{(l+2)}\mathcal{X}_1, e^{\nu_c}r^{(l-1)}\omega^{-2}\mathcal{X}_2, 0, 0], \qquad \mathcal{Y}_i^{(2)} = [0,0,0,1] \tag{89}$$

which guarantees (using the freedom to rescale the fluid solution) that the linear combination $\mathcal{Y}_i^{(\text{tot})} = \mathcal{C}_1\mathcal{Y}_i^{(1)} + \mathcal{C}_2\mathcal{Y}_i^{(2)}$ satisfies the boundary conditions at the surface for any values of $\mathcal{C}_i$. At the surface we arbitrarily fix the overall normalisation of the solution by setting $\tilde{\mathcal{Y}}_1 = 1$ where we will use a tilde to distinguish the outer crust solution from the inner. From the vacuum value of the phase function (66) we then obtain the boundary value of $\tilde{\mathcal{Y}}_2$,

$$\tilde{\mathcal{Y}}_2 = \frac{i\omega R - (R - 2M)(R^{-1} + g_s)}{R^3\omega^2} \tag{90}$$

In addition we require that $\tilde{\mathcal{Y}}_3 = 0$. Thus, we choose the independent solutions via the interface values

$$\tilde{\mathcal{Y}}_i^{(1)} = [1, \tilde{\mathcal{Y}}_2(g_s), 0, 0], \qquad \tilde{\mathcal{Y}}_i^{(2)} = [0,0,0,1] \tag{91}$$

This again guarantees that the linear combination $\tilde{\mathcal{Y}}_i^{(\text{tot})} = \tilde{\mathcal{Y}}_i^{(1)} + \mathcal{C}_3\tilde{\mathcal{Y}}_i^{(2)}$ satisfies the boundary conditions at the interface for any value of $\mathcal{C}_3$. Note that we only have a single coefficient in the linear combination due to the choice of normalisation.

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[6] The solution will be unique only if one assumes that $\mathcal{Y}_4$ is continuous in the crust. This would seem natural, but is in fact not required by the equations. One could imagine discontinuities at *e.g.* phase transitions between different regions in the solid. We take $\mathcal{Y}_4$ to be continuous, corresponding to the assumption that the crust behaves as a single solid. It is our understanding that the sharp discontinuities found in calculations of nuclear matter at absolute zero temperature are likely to be smoothed in a real neutron star. At least on the length-scales of interest in a global mode analysis. Moreover we find it highly unlikely that, even if such discontinuities appear, the layers will slip freely along the boundaries.

We have now ensured that the boundary conditions are fulfilled everywhere except at the matching point. Here we require that all $\mathcal{Y}_i$ are continuous. Thus we demand that

$$\mathcal{C}_1 \mathcal{Y}_i^{(1)} + \mathcal{C}_2 \mathcal{Y}_i^{(2)} = \tilde{\mathcal{Y}}_i^{(1)} + \mathcal{C}_3 \tilde{\mathcal{Y}}_i^{(2)} \tag{92}$$

This condition can be put in matrix form as

$$\mathbf{M}\bar{v} = 0 \tag{93}$$

where

$$\mathbf{M} = \begin{pmatrix} \mathcal{Y}_1^{(1)} & \mathcal{Y}_1^{(2)} & \tilde{\mathcal{Y}}_1^{(1)} & \tilde{\mathcal{Y}}_1^{(2)} \\ \mathcal{Y}_2^{(1)} & \mathcal{Y}_2^{(2)} & \tilde{\mathcal{Y}}_2^{(1)} & \tilde{\mathcal{Y}}_2^{(2)} \\ \mathcal{Y}_3^{(1)} & \mathcal{Y}_3^{(2)} & \tilde{\mathcal{Y}}_3^{(1)} & \tilde{\mathcal{Y}}_3^{(2)} \\ \mathcal{Y}_4^{(1)} & \mathcal{Y}_4^{(2)} & \tilde{\mathcal{Y}}_4^{(1)} & \tilde{\mathcal{Y}}_4^{(2)} \end{pmatrix}$$

and

$$\bar{v} = [\mathcal{C}_1, \mathcal{C}_2, -1, -\mathcal{C}_3]^T \tag{94}$$

In order for a solution exist we need $\det(\mathbf{M}) = \det[\mathbf{M}(\omega)] = 0$ which is the eigenvalue equation that gives the quasi-normal frequencies. We solve the mode condition using a standard complex root finder based on Müller's method.

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